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Scattering of water waves by axisymmetric bodies in a channel

P. McIVER[1] and G.S. BENNETT[2]

[1]*Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, Leics. LE11 3TU, UK*
[2]*Department of Mathematics and Statistics, Brunel University, Uxbridge, Middx UB8 3PH, UK*

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Abstract. Methods are presented for the calculation of wave forces on a vertically axisymmetric body arbitrarily placed within a channel. Integral representations of singular solutions of the Helmholtz equation, called channel multipoles here, are derived and these allow straightforward solution of the scattering problem for a vertical cylinder extending throughout the depth. In contrast to previous methods there is no need to sum series of images. These multipoles are also used in deriving an approximate solution valid when the radius of the cylinder is small relative to the wavelength and channel width.

To solve for arbitrary shaped axisymmetric bodies, a plane-wave approximation is developed based on the assumption that the wavelength is much less than the channel width. Comparisons with the accurate solution for a vertical cylinder suggest that this approximate method performs well even when this assumption is clearly violated. The results of calculations of wave forces on a truncated cylinder are also given.

All of the methods described may be applied just as easily to the case of an off-centre body as to a centrally-placed body.

1. Introduction

To assist in the interpretation of results from the wave-tank testing of offshore structures there is a need to quantify the effects of the tank walls. Perhaps the simplest geometry that has been investigated is that of a vertical cylinder extending throughout the uniform depth of a parallel-walled channel. On the basis of the linearised theory of water waves, the depth dependence may be removed and the flow is found to be described by a solution of the two-dimensional Helmholtz equation satisfying appropriate boundary and radiation conditions. This is equivalent to a problem in the theory of electromagnetic waveguides, and the case of the cylinder being centrally placed between the walls has received considerable attention in the literature. For references to this body of work and its relation to the water-wave problem, see Martin and Dalrymple [1] and Thomas [2]. In the context of water waves, solutions of the scattering problem valid for any length of incident wave have been given by Spring and Monkmeyer [3], using the method of images, and Thomas [2], using an integral equation approach. The approach of Spring and Monkmeyer has been extended to the scattering and radiation of waves by a truncated vertical cylinder by Yeung and Sphaier [4, 5]. A feature of the work in references [2–5] is the need to sum slowly convergent series of Hankel functions. Yeung and Sphaier [4] give a careful treatment of such series which facilitates their accurate summation. In the present work, such series are, in effect, replaced by integral representations which may be evaluated without much difficulty.

The method presented here is new in the context of channel scattering problems although the basis of the idea is familiar in water waves. For example, Ursell [6] solved the two-dimensional problem of scattering by a submerged, circular cylinder by constructing the solution from an infinite set of multipole potentials. Each multipole individually satisfied all the conditions of the problem except the boundary condition on the cylinder surface. Ursell expressed the solution as a sum over the multipoles, with unknown coefficients, and imposed the final boundary condition to give an infinite system for those coefficients which is readily solved by truncation. Subsequently, Thorne [7] derived a range of multipoles in two and three dimensions that allow the straightforward solution of many problems. Following the procedure used by Thorne, 'channel multipoles' are derived in Section 3 of the present paper. Each multipole is a singular solution of the two-dimensional Helmholtz equation that satisfies the boundary condition of no flow through the walls and the radiation conditions of outgoing waves at large distances along the channel. A restricted set of these multipoles has already been used by Callan, Linton and Evans [8] and McIver [9] in constructing solutions for waves trapped by a vertical cylinder on the centre line of a channel. In Section 4, the full set of multipoles are used to solve the scattering problem for a vertical cylinder arbitrarily placed within the channel and extensive results are presented for the wave forces on such a cylinder. In concurrent work, Linton and Evans [10] have used channel multipoles to investigate the scattering and radiation of waves by a vertical cylinder on the centre line of a channel.

Using these multipoles it is straightforward to construct new approximate solutions to the channel scattering problem. In Section 5, a solution valid when the wavelength is much greater than the cylinder radius is derived using the method of matched asymptotic expansions building on the ideas used in water-wave scattering problems by Davis and Leppington [11] and in channel problems by McIver [9].

Another useful approximation is based on the assumption that the wavelength is much less than the channel width. This is a wide-spacing approximation in the sense that all of the images in the channel walls are widely spaced on the scale of the wavelength so that, to a first approximation, the waves scattered by each image appear as plane waves to the remaining images. This 'plane-wave' approximation was introduced by Simon [12] to investigate the performance of finite arrays of wave-energy devices. The method was extended to include non-plane correction terms and applied to the scattering by arrays of vertical cylinders by McIver and Evans [13] and to radiation within arrays of truncated cylinders by McIver [14]. Simon [15] extended his work to an infinite line of bodies, equivalent to a body in a channel, and preliminary results for a vertical cylinder are given by McIver and Simon [16]. In both of those works, the results were found by summing over the images. Here a more direct approach is used, in Section 6 approximations to the channel multipoles are found, including the correction terms introduced by McIver and Evans [13]. In Section 7, these are used to solve the scattering problem for a truncated vertical cylinder using the (assumed known) scattering coefficients for open water.

The results of calculations using all of the above methods are collected together and compared in Section 8. Both of the approximate methods are shown to perform well when compared with the accurate solution.

2. The vertical cylinder: formulation

An infinitely long channel of uniform depth h has parallel walls a distance $2b$ apart. Cartesian coordinates are chosen with the origin in the mean free surface and midway between the channel walls so that the x-axis is directed along the channel and the z-axis vertically upwards. A vertical, circular cylinder of radius a extends throughout the depth and has its axis at $(x, y) = (0, d)$. Plane polar coordinates r and θ are defined by

$$x = r\cos\theta \text{ and } y - d = r\sin\theta\,. \tag{2.1}$$

The full geometry is illustrated in Fig. 1.

A plane wave of amplitude A and frequency ω is incident from large negative x. Under the usual assumptions of linear water-wave theory the time-harmonic flow may be described by a velocity potential

$$\Phi(x, y, z, t) = \Re\left\{ -\frac{igA\cosh k(z+h)}{\omega\cosh kh}\,\phi_T(x, y)\,\mathrm{e}^{-i\omega t}\right\}, \tag{2.2}$$

where k is the wavenumber satisfying the dispersion relation

$$\omega^2 = gk\tanh kh\,, \tag{2.3}$$

g is the acceleration due to gravity and $\Re$ indicates that the real part is to be taken. The form of the potential has been chosen to satisfy the linearised free-surface condition, on $z = 0$, and the bed condition, on $z = -h$. The potential Φ must satisfy the three-dimensional Laplace equation so that, on substituting the form (2.2), the complex-valued function $\phi_T(x, y)$ may be seen to satisfy the Helmholtz equation. The incident wave is described by

$$\phi_I = \mathrm{e}^{ikx} = \mathrm{e}^{ikr\cos\theta} = \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(kr)\cos n\theta \tag{2.4}$$

(Abramowitz and Stegun [17], equations (9.1.44) and (9.1.45)), where

$$\varepsilon_0 = 1,\ \varepsilon_n = 2\ n \geqslant 1 \tag{2.5}$$

and J_n is a Bessel function. The total potential ϕ_T is decomposed as

$$\phi_T = \phi_I + \phi\,. \tag{2.6}$$

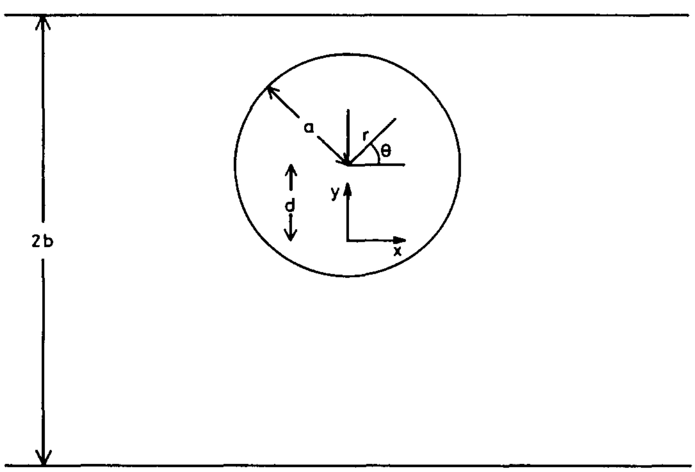


Fig. 1. Definition sketch.

The potential ϕ describing the scattered wave field will then satisfy the Helmholtz equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0 \tag{2.7}$$

in the fluid domain, the conditions of no flow through the channel walls,

$$\frac{\partial \phi}{\partial y} = 0 \text{ on } y = \pm b , \tag{2.8}$$

and the cylinder surface,

$$\frac{\partial \phi}{\partial r} = -\frac{\partial \phi_I}{\partial r} \text{ on } r = a , \tag{2.9}$$

and the radiation condition specifying that the scattered waves must be outgoing as $|x| \to \infty$.

The solution for ϕ is expressed in terms of channel multipoles, derived in the following section.

3. Derivation of channel multipoles

The channel multipoles derived here are singular solutions of the Helmholtz equation (2.7) that, by construction, satisfy the conditions (2.8) on the channel wall and the radiation condition. Each multipole is singular at the point $(x, y) = (0, d)$, that is at $r = 0$ where r is a polar coordinate as defined in (2.1). Two sets of multipoles are defined. The first, denoted by ϕ_n, have a singular part $H_n(kr) \cos n\theta$ and the second set, denoted by ψ_n, have a singular part $H_n(kr) \sin n\theta$. Here, H_n denotes the Hankel function of the first kind and order n.

The basis of the derivation is the integral representation

$$H_n(kr)\, \mathrm{e}^{in\theta} = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ikxt-\gamma k(y-d)}}{\gamma}\, \mathrm{e}^{in \sin^{-1} t}\, \mathrm{d}t , \; y > d , \tag{3.1}$$

where $\gamma = (t^2 - 1)^{1/2} = -i(1 - t^2)^{1/2}$, which follows from Twersky [18, equation (31)] after a rotation of the axes through $\pi/2$. The path of integration runs beneath the branch point at $t = 1$ and above that at $t = -1$. Define τ by

$$\cosh \tau = t , \; \sinh \tau = \gamma . \tag{3.2}$$

By combining (3.1) with a similar expression where n is replaced by $-n$ and considering the cases of odd and even n separately, it is easy to show that

$$H_n(kr) \cos n\theta = \frac{i^{n-1}}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ikxt-\gamma k|y-d|}}{\gamma} \cosh n\tau\, \mathrm{d}t \tag{3.3}$$

and

$$H_n(kr) \sin n\theta = -\mathrm{sgn}(y - d) \frac{i^n}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-ikxt-\gamma k|y-d|}}{\gamma} \sinh n\tau\, \mathrm{d}t , \tag{3.4}$$

where the results

$$\begin{aligned}&\cos(2n\sin^{-1}t)=(-1)^n\cosh 2n\tau,\ \cos((2n+1)\sin^{-1}t)=i(-1)^n\sinh(2n+1)\tau\,,\\&\sin(2n\sin^{-1}t)=-i(-1)^n\sinh 2n\tau,\ \sin((2n+1)\sin^{-1}t)=(-1)^n\cosh(2n+1)\tau\end{aligned}\tag{3.5}$$

have been used.

Channel multipoles are constructed following the method of Thorne [7]. For the multipoles ϕ_n corresponding to (3.3) write

$$\phi_n=H_n(kr)\cos n\theta+\frac{i^{n-1}}{\pi}\int_{-\infty}^{\infty}(A(t)\,\mathrm{e}^{\gamma k(y-d)}+B(t)\,\mathrm{e}^{-\gamma k(y-d)})\,\frac{\mathrm{e}^{-ikxt}}{\gamma}\cosh n\tau\,\mathrm{d}t\,.\tag{3.6}$$

The boundary conditions (2.8) of no flow through the channel walls at $y=\pm b$ give simultaneous equations for the unknown functions $A(t)$ and $B(t)$ which, when solved, lead to

$$\begin{aligned}\phi_n&=H_n(kr)\cos n\theta\\&\quad+\frac{i^{n-1}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\gamma k(y-d)}(\mathrm{e}^{-2\gamma\mu}+\mathrm{e}^{2\gamma\nu})+\mathrm{e}^{-\gamma k(y-d)}(\mathrm{e}^{-2\gamma\mu}+\mathrm{e}^{-2\gamma\nu})}{\gamma\sinh 2\gamma\mu}\,\mathrm{e}^{-ikxt}\cosh n\tau\,\mathrm{d}t\,,\end{aligned}\tag{3.7}$$

where and $\mu=kb$ and $\nu=kd$. If the integral representation (3.3) is introduced into (3.7), then this may be written

$$\phi_n=\frac{i^{n-1}}{\pi}\int_{-\infty}^{\infty}\frac{\cosh\gamma(k|y-d|-2\mu)+\cosh\gamma k(y+d)}{\gamma\sinh 2\gamma\mu}\,\mathrm{e}^{-ikxt}\cosh n\tau\,\mathrm{d}t\,.\tag{3.8}$$

In general, the integrands in equations (3.7) and (3.8) have poles at the solutions of $2\gamma\mu=\pm im\pi$, where m is a non-negative integer. The corresponding values of t are $\pm t_m$ where

$$t_m=(1-(m\pi/2\mu)^2)^{1/2}\,,\ m=0,1,2,\dots M\tag{3.9}$$

$$t_m=i((m\pi/2\mu)^2-1)^{1/2}\,,\ m\geq M+1\,,\tag{3.10}$$

and M is the integer satisfying

$$M\pi<2\mu<(M+1)\pi\,.\tag{3.11}$$

The $M+1$ poles given by (3.9) lie on the real t axis and give rise to propagating waves as $|x|\to\infty$. To obtain only outgoing waves the integration path must run beneath the poles on the positive real axis and above the poles on the negative real axis. Applying the residue theorem to evaluate the integral in (3.8) gives the alternative representation

$$\begin{aligned}\phi_n=\frac{(-i\,\mathrm{sgn}\,x)^n}{\mu}&\left\{\mathrm{e}^{ik|x|}+\sum_{m=1}^{\infty}\left[\cos\left(\frac{m\pi}{2\mu}k(y-d)\right)\right.\right.\\&\left.\left.+(-1)^m\cos\left(\frac{m\pi}{2\mu}k(y+d)\right)\right]t_m^{-1}\,\mathrm{e}^{ik|x|t_m}\cosh n\tau_m\right\},\end{aligned}\tag{3.12}$$

where

$$\cosh \tau_m = t_m \, . \tag{3.13}$$

For $m \geq M+1$, t_m is imaginary and the corresponding terms in the above summation decay exponentially as $|x| \to \infty$ so that, for a given μ, there are only a finite number (≥ 1) of propagating modes. If $\nu = kd = 0$ so that the singular point is on the centre line of the channel then only even values of m contribute to the summation and all modes are symmetric in y.

Channel multipoles ψ_n corresponding to (3.4) are constructed in the same way and the result is

$$\psi_n = H_n(kr) \sin n\theta + \frac{i^n}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\gamma k(y-d)}(e^{-2\gamma\mu} - e^{2\gamma\nu}) - e^{-\gamma k(y-d)}(e^{-2\gamma\mu} - e^{-2\gamma\nu})}{\gamma \sinh 2\gamma\mu} e^{-ikxt} \sinh n\tau \, dt \, . \tag{3.14}$$

The location of the poles and the choice of contour is the same as for ϕ_n above with the exception that there are no longer poles at $t = \pm 1$. The representations corresponding to (3.8) and (3.12) are

$$\psi_n = \frac{i^n}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(y-d) \sinh \gamma(k|y-d| - 2\mu) - \sinh \gamma k(y+d)}{\gamma \sinh 2\gamma\mu} e^{-ikxt} \sinh n\tau \, dt \tag{3.15}$$

and

$$\psi_n = \frac{(-i \operatorname{sgn} x)^{n+1}}{\mu i} \sum_{m=1}^{\infty} \left[\sin\left(\frac{m\pi}{2\mu} k(y-d)\right) - (-1)^m \sin\left(\frac{m\pi}{2\mu} k(y+d)\right) \right] t_m^{-1} e^{ik|x|t_m} \sinh n\tau_m \, . \tag{3.16}$$

If $\nu = kd = 0$ only the odd values of m contribute to the summation and all the modes are antisymmetric in y. In addition, if $\mu < \pi/2$ then all t_m, with m odd, are pure imaginary and there is no radiation of energy to infinity along the channel. These non-radiating antisymmetric multipoles were used by Callan, Linton and Evans [8] and McIver [9] to construct trapped wave solutions for a cylinder on the centre-line of a channel.

For the solution of the cylinder scattering problem it is necessary to expand the multipole potentials in terms of the polar coordinates r and θ. The identity

$$e^{\frac{1}{2}Z(T-T^{-1})} = \sum_{m=-\infty}^{\infty} T^m J_m(Z) \tag{3.17}$$

(Abramowitz and Stegun [17], equation (9.1.41)) is used, where J_m is the Bessel function of order m. Making the substitutions $Z = kr$ and $T = -ie^{\tau \pm i\theta}$ gives

$$e^{-ikxt \pm \gamma k(y-d)} = \sum_{m=0}^{\infty} \varepsilon_m (-i)^m (\cosh m\tau \cos m\theta \pm i \sinh m\tau \sin m\theta) J_m(kr) \, , \tag{3.18}$$

where ε_m is defined by equation (2.5). Inserting (3.18) into (3.7) and (3.14) gives

$$\phi_n = H_n(kr)\cos n\theta + \sum_{m=0}^{\infty} (\alpha_{nm}\cos m\theta + \beta_{nm}\sin m\theta)J_m(kr)\,, \tag{3.19}$$

where

$$\alpha_{nm} = \frac{\varepsilon_m(-i)^{m-n+1}}{\pi}\int_{-\infty}^{\infty} \frac{e^{-2\gamma\mu} + \cosh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\cosh m\tau\cosh n\tau\,dt\,, \tag{3.20}$$

$$\beta_{nm} = \frac{2(-i)^{m-n}}{\pi}\int_{-\infty}^{\infty} \frac{\sinh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\sinh m\tau\cosh n\tau\,dt\,, \tag{3.21}$$

and

$$\psi_n = H_n(kr)\sin n\theta + \sum_{m=0}^{\infty} (a_{nm}\cos m\theta + b_{nm}\sin m\theta)J_m(kr)\,, \tag{3.22}$$

where

$$a_{nm} = -\frac{\varepsilon_m(-i)^{m-n}}{\pi}\int_{-\infty}^{\infty} \frac{\sinh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\cosh m\tau\sinh n\tau\,dt\,, \tag{3.23}$$

$$b_{nm} = \frac{2(-i)^{m-n-1}}{\pi}\int_{-\infty}^{\infty} \frac{e^{-2\gamma\mu} - \cosh 2\gamma\nu}{\gamma\sinh 2\gamma\mu}\sinh m\tau\sinh n\tau\,dt\,. \tag{3.24}$$

Note that $\beta_{n0} = b_{n0} = 0$ for all n. The expansions (3.19) and (3.22) are valid for $0 < r < 2(b-d)$.

The coefficients defined by (3.20–21) and (3.23–24) will be zero whenever the integrands are odd functions of t. From equation (3.5) it may be observed that $\cosh 2n\tau$ and $\sinh(2n+1)\tau$ are even functions of t while $\sinh 2n\tau$ and $\cosh(2n+1)\tau$ are odd functions of t. Thus, α_{nm} and b_{nm} are zero if $m+n$ is odd while β_{nm} and a_{nm} are zero if $m+n$ is even. To simplify the presentation of the solution to the scattering problem the full expansions in (3.19) and (3.22) will be retained.

4. The vertical cylinder: full solution

The scattered wave field is expressed as a sum of channel multipoles in the form

$$\phi = \sum_{n=0}^{\infty} \varepsilon_n i^n (A_n\phi_n + B_n\psi_n)\,. \tag{4.1}$$

There is no multiple ψ_0, but to obtain a compact notation it has been included in (4.1) with the implicit assumption that $B_0 = 0$. By the construction of the multipoles, equation (4.1) satisfies all the conditions of the problem except the body boundary condition. Substituting the polar coordinate expansions (3.19) and (3.22) into (4.1) and interchanging the order of the summations gives

$$\phi = \sum_{n=0}^{\infty} \left\{ \varepsilon_n i^n (A_n \cos n\theta + B_n \sin n\theta) H_n(kr) \right.$$
$$\left. + \left[\cos n\theta \sum_{p=0}^{\infty} \varepsilon_p i^p (A_p \alpha_{pn} + B_p a_{pn}) + \sin n\theta \sum_{p=0}^{\infty} \varepsilon_p i^p (A_p \beta_{pn} + B_p b_{pn}) \right] J_n(kr) \right\} . \tag{4.2}$$

Applying the body boundary condition (2.9) and using the orthogonality over $(0, 2\pi)$ of the functions $\{\cos n\theta,\ n \geqslant 0\}$ and $\{\sin n\theta,\ n \geqslant 1\}$, gives the two sets of simultaneous equations

$$A_n \varepsilon_n i^n Q_n + \sum_{p=0}^{\infty} \varepsilon_p i^p (A_p \alpha_{pn} + B_p a_{pn}) = -\varepsilon_n i^n \ , \ n \geqslant 0 \tag{4.3}$$

and

$$B_n \varepsilon_n i^n Q_n + \sum_{p=0}^{\infty} \varepsilon_p i^p (A_p \beta_{pn} + B_p b_{pn}) = 0 \ , \ n \geqslant 1 \ , \tag{4.4}$$

where

$$Q_n = \frac{H_n'(ka)}{J_n'(ka)} \ . \tag{4.5}$$

The systems (4.3) and (4.4) may be solved by truncation.

Following Linton and Evans [19], the summations over p in (4.3) and (4.4) may be substituted back into (4.2) to obtain a simplified form for the potential on or near the cylinder. Thus

$$\phi = \sum_{n=0}^{\infty} \varepsilon_n i^n (A_n \cos n\theta + B_n \sin n\theta)(H_n(kr) - Q_n J_n(kr)) - \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(kr) \cos n\theta \ . \tag{4.6}$$

The final term in (4.6) is just the incident wave (2.4). When evaluated on the cylinder $r = a$ then

$$H_n(ka) - Q_n J_n(ka) = -\frac{2i}{\pi ka J_n'(ka)} \ , \tag{4.7}$$

where a Wronskian relation for Bessel functions (Abramowitz and Stegun [17], equation (9.1.16)) has been used. Thus the total potential evaluated on the cylinder surface is

$$\phi_T = \phi + \phi_I = -\frac{2i}{\pi ka} \sum_{n=0}^{\infty} \frac{\varepsilon_n i^n}{J_n'(ka)} (A_n \cos n\theta + B_n \sin n\theta) \ . \tag{4.8}$$

The first-order forces on the cylinder are found by integrating the pressure over the surface. The subscript j is used to denote either the x or y component of the force. The pressure is given in terms of the potential Φ, defined in equation (2.2), by the linearised Bernoulli equation and so the horizontal components of the force are $\Re\{f_j \, e^{-i\omega t}\}$ where

$$f_j = -\frac{\rho g a A}{k} \tanh kh \int_0^{2\pi} \phi_T|_{r=a} \, n_j \, d\theta \tag{4.9}$$

and n_j is the appropriate component of the outward normal to the cylinder surface ($n_x = \cos\theta$, $n_y = \sin\theta$). Substitution of the total potential given in equation (4.8) yields

$$\left.\begin{matrix} f_x \\ f_y \end{matrix}\right\} = -\frac{4\rho g A \tanh kh}{k^2 J_1'(ka)} \left\{\begin{matrix} A_1 \\ B_1 \end{matrix}\right. . \tag{4.10}$$

If the cylinder is in open water the in-line force is

$$f^{(o)} = \frac{4\rho g A \tanh kh}{k^2 H_1'(ka)} \tag{4.11}$$

(MacCamy and Fuchs [20]) and, so relative to the open water case, the forces are

$$F_x = \frac{f_x}{f^{(o)}} = -Q_1 A_1 \text{ and } F_y \equiv \frac{f_y}{f^{(o)}} = -Q_1 B_1 . \tag{4.12}$$

5. The vertical cylinder: small radius solution

Here an approximate solution to the cylinder scattering problem, formulated in Section 2, is derived under the assumptions that the radius of the cylinder is small relative to the wavelength, that is $\varepsilon \equiv ka \ll 1$, and that the wavelength and channel width are of the same order of magnitude, that is $\mu \equiv kb = O(1)$. A consequence of these assumptions is that the cylinder radius is much less than the channel width, that is $a \ll b$. For simplicity, it will also be assumed that the cylinder is not close to one of the channel walls, so that $|d| \ll b$, but a solution without this restriction could be derived along similar lines. The solution is by the method of matched asymptotic expansions and requires that the flow domain be divided into two regions. The inner region surrounds the cylinder to distances $r \ll k^{-1}$ and the outer region is external to this at distances $r \gg a$. Solutions containing unknown constants are constructed separately in the two regions and, with the assumption $ka \ll 1$, they can be fully determined by matching in an overlap region.

For the inner region, around the cylinder, define scaled coordinates with origin at the cylinder axis by

$$\xi = \frac{x}{a}\,, \; \eta = \frac{y-d}{a}\,, \; \varrho = \frac{r}{a}\,. \tag{5.1}$$

The inner scattered potential $\Psi_i(\xi, \eta) \equiv \phi(x, y)$ satisfies the field equation (2.7) and the body boundary condition (2.9) which, in terms of the inner coordinates, are

$$\nabla^2\Psi_i + \varepsilon^2\Psi_i = 0 \tag{5.2}$$

and

$$\frac{\partial \Psi_i}{\partial \varrho} = -\frac{\partial}{\partial \varrho}\left(e^{i\varepsilon\xi}\right) = -i\varepsilon\cos\theta + \varepsilon^2\cos^2\theta + \frac{1}{2}\, i\varepsilon^3\cos^3\theta + O(\varepsilon^4) \text{ on } \varrho = 1 \tag{5.3}$$

respectively.

Scaled outer coordinates are defined by

$$X = kx\,, \; Y = k(y-d)\,, \; R = kr\,. \tag{5.4}$$

The outer region solution $\Psi_o(X, Y) \equiv \phi(x, y)$ must satisfy all the conditions of the problem except for the body boundary condition (2.9). As with the full solution, the outer solution will be constructed from the channel multipoles derived in Section 3 but now expressed in terms of the outer variables (5.4).

From (5.2), the leading order inner solution for $\varepsilon \ll 1$ is harmonic and the body boundary condition (5.3) is satisfied to leading order by

$$\Psi_i^{(1)} = \varepsilon i \frac{\cos\theta}{\varrho}, \tag{5.5}$$

where $\Psi_i^{(l)}$ denotes the inner solution to order l in ε. Strictly, possible solutions of the homogeneous problem should be allowed for here and at lower orders, but these cannot be matched with the leading order inner expansion of any of the (singular) multipoles and so are omitted. When expressed in outer coordinates (5.5) gives

$$\Psi_i^{(1,2)} = \varepsilon^2 i \frac{\cos\theta}{R}, \tag{5.6}$$

where $\Psi_i^{(l,m)}$ denotes the inner solution to order l in ε rewritten in terms of the outer variables and expanded to order m. A similar notation is used for the outer potential Ψ_o. Thus $\Psi_o^{(m)}$ denotes the order m outer solution which when rewritten in terms of the inner variables and expanded to order l is denoted by $\Psi_o^{(m,l)}$. The matching principle requires $\Psi_i^{(l,m)} \equiv \Psi_o^{(m,l)}$ (see, for example, Crighton and Leppington [21]).

The outer expansion (5.6) indicates that the leading-order outer solution is at $O(\varepsilon^2)$, will contain terms no more singular than R^{-1} and have no dipole term with a $\sin\theta$ dependence. Thus

$$\Psi_o^{(2)} = \varepsilon^2\{A_0\phi_0 + A_1\phi_1\}, \tag{5.7}$$

where the ϕ_n are the channel multipoles defined by equation (3.7) and A_0 and A_1 are constants to be found from the matching. It follows from the expansion of ϕ_n in terms of polar coordinates, equation (3.19), that the inner expansion of (5.7) is

$$\Psi_o^{(2,2)} = \varepsilon^2\left\{A_0\left[1 + \frac{2i}{\pi}\left(\ln\frac{\varepsilon\varrho}{2} + C\right) + \alpha_{00}\right] + A_1\left[\frac{2}{\pi i}\frac{\cos\theta}{\varepsilon\varrho}\right]\right\}, \tag{5.8}$$

where the required expansions of Bessel functions are given by Abramowitz and Stegun [17, p. 360] and C is Euler's constant. From (5.8), $\Psi_o^{(2,2)}$ contains terms at orders ε, $\varepsilon^2\ln\varepsilon$ and ε^2. To be able to match with this the inner solution must have the form

$$\Psi_i^{(2)} = \varepsilon i \frac{\cos\theta}{\varrho} + \varepsilon^2 \ln\varepsilon\, \Psi_{i,21} + \varepsilon^2\Psi_{i,2}. \tag{5.9}$$

This inner solution is substituted into equations (5.2–3), and the coefficients of like gauge functions in ε equated. This shows that $\Psi_{i,21}$ is a harmonic function satisfying a homogeneous body boundary condition. Possible solutions are of the form

$$(\varrho^n + \varrho^{-n})\begin{cases}\cos n\theta \\ \sin n\theta\end{cases}, \quad n = 0, 1, 2\ldots \tag{5.10}$$

but, apart from constants, these solutions grow too fast as $\varrho \to \infty$ to be matched with (5.8).

Also, $\Psi_{i,2}$ is a harmonic function satisfying the body boundary condition

$$\frac{\partial \Psi_{i,2}}{\partial \varrho} = \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \text{ on } \varrho = 1 \tag{5.11}$$

and a particular solution is

$$\Psi_{i,2} = \frac{1}{2}\left(\ln \varrho - \frac{\cos 2\theta}{2\varrho^2}\right). \tag{5.12}$$

To this may be added the solutions, given in (5.10), of the homogeneous problem but again only constants are suitable. Thus,

$$\Psi_i^{(2)} = \varepsilon i \frac{\cos\theta}{\varrho} + \varepsilon^2 \ln \varepsilon\, B_0 + \varepsilon^2\left\{B_1 + \frac{1}{2}\ln \varrho - \frac{\cos 2\theta}{4\varrho^2}\right\} \tag{5.13}$$

which has an outer expansion

$$\Psi_i^{(2,2)} = \varepsilon i \frac{\cos\theta}{\varrho} + \varepsilon^2 \ln \varepsilon\, B_0 + \varepsilon^2\left\{B_1 + \frac{1}{2}\ln \varrho\right\} \tag{5.14}$$

when written in terms of inner coordinates. Matching (5.14) with (5.8) gives

$$A_1 = -\frac{\pi}{2}, \; A_0 = -\frac{\pi i}{4}, \; B_0 = \frac{2i}{\pi}, \; A_0 = \frac{1}{2},$$

$$B_1 = A_0\left(1 + \frac{2i}{\pi}(C - \ln 2) + \alpha_{00}\right). \tag{5.15}$$

So, from (5.7), the leading-order outer solution is

$$\Psi_o^{(2)} = -\varepsilon^2 \frac{\pi}{2}\left\{\frac{i}{2}\phi_0 + \phi_1\right\}. \tag{5.16}$$

(Using the first term in the expansion (3.12) of ϕ_n, approximations to the reflection and transmission coefficients, $-\varepsilon^2 3\pi i/4\mu$ and $1 + \varepsilon^2 \pi i/4\mu$ respectively, may be obtained in agreement with Twersky [18].)

The correct development of the inner solution may be deduced from the inner expansion of (5.16),

$$\begin{aligned}\Psi_o^{(2,3)} = -\varepsilon^2 \frac{\pi}{2}\Bigg\{&\frac{i}{2}\left[1 + \frac{2i}{\pi}\left(\ln\frac{\varepsilon\varrho}{2} + C\right) + \alpha_{00} + \beta_{01}\sin\theta\,\frac{1}{2}\varepsilon\varrho\right] \\ &+ \left[\frac{2}{\pi i}\frac{1}{\varepsilon\varrho} + \frac{i}{\pi}\varepsilon\varrho\ln\varepsilon\varrho + \frac{1}{2}\varepsilon\varrho\left(1 - \frac{i}{\pi}(1 - 2C + 2\ln 2) + \alpha_{11}\right)\right]\cos\theta\Bigg\}.\end{aligned} \tag{5.17}$$

In addition to those that have already arisen in $\Psi_o^{(2,2)}$, equation (5.8), there are terms at orders $\varepsilon^3 \ln \varepsilon$ and ε^3 and this must be reflected in the inner solution. From equations (5.2–3), the term at $O(\varepsilon^3 \ln \varepsilon)$ in the inner solution will be harmonic and satisfy a homogeneous condition while the $O(\varepsilon^3)$ inner potential will satisfy a non-homogeneous field equation and boundary condition. Hence, the inner solution continues as

$$\Psi_i^{(3)} = \Psi_i^{(2)} + \varepsilon^3 \ln \varepsilon\{B_2 + B_3(\varrho + \varrho^{-1})\cos\theta\} + \varepsilon^3 \Psi_{i,3}, \tag{5.18}$$

where the homogeneous solutions at $O(\varepsilon^3 \ln \varepsilon)$ have been chosen to allow matching with (5.17) and, from (5.2), Ψ_{i3} satisfies

$$\nabla^2 \Psi_{i,3} = -i \frac{\cos\theta}{\varrho} \tag{5.19}$$

in the fluid and, from (5.3), the boundary condition

$$\frac{\partial \Psi_{i,3}}{\partial \varrho} = \frac{i}{2} \cos^3\theta = \frac{i}{8}\left(\cos 3\theta + 3\cos\theta\right) \text{ on } \varrho = 1 . \tag{5.20}$$

A particular solution of (5.19–20), together with suitable homogeneous solutions to allow matching with (5.17), is

$$\Psi_{i,3} = -\frac{i}{2}\left(\varrho \ln \varrho \cos\theta + \frac{1}{12}\frac{\cos 3\theta}{\varrho^3} + \frac{7}{4}\frac{\cos\theta}{\varrho}\right) + B_4 + (\varrho + \varrho^{-1})(B_5 \cos\theta + B_6 \sin\theta) \tag{5.21}$$

and so the inner expansion of $\Psi_i^{(3)}$ is

$$\begin{aligned}\Psi_i^{(3,2)} = {} & \varepsilon i \frac{\cos\theta}{\varrho} + \frac{1}{2}\varepsilon^2 \ln\varepsilon + \varepsilon^2\left\{B_1 + \frac{1}{2}\ln\varrho\right\} + \varepsilon^3 \ln\varepsilon \{B_2 + B_3 \varrho \cos\theta\} \\ & + \varepsilon^3\left\{-\frac{i}{2}\varrho \ln\varrho \cos\theta + B_5 \varrho \cos\theta + B_6 \varrho \sin\theta\right\} .\end{aligned} \tag{5.22}$$

Matching (5.22) with the inner expansion of the outer solution (5.17) gives

$$B_2 = 0 , \; B_3 = -\frac{i}{2} , \; B_5 = -\frac{\pi}{4}\left(1 - \frac{i}{\pi}(1 - 2C + 2\ln 2) + \alpha_{11}\right) , \; B_6 = -\frac{\pi i}{8}\beta_{01} . \tag{5.23}$$

To summarise, the leading-order outer solution is given by (5.16) and the inner solution to $O(\varepsilon^3)$ is

$$\begin{aligned}\Psi_i^{(3)} = {} & \varepsilon i \frac{\cos\theta}{\varrho} + \frac{1}{2}\varepsilon^2 \ln\varepsilon + \varepsilon^2\left\{B_1 + \frac{1}{2}\ln\varrho - \frac{\cos 2\theta}{4\varrho^2}\right\} - \frac{i}{2}\varepsilon^3 \ln\varepsilon\,(\varrho + \varrho^{-1})\cos\theta \\ & + \varepsilon^3\left\{-\frac{i}{2}\varrho \ln\varrho \cos\theta - \frac{i}{24}\frac{\cos 3\theta}{\varrho^3} - \frac{7i}{8}\frac{\cos\theta}{\varrho} + B_4 + (\varrho + \varrho^{-1}) \right. \\ & \left. \times (B_5 \cos\theta + B_6 \sin\theta)\right\} ,\end{aligned} \tag{5.24}$$

where B_1 is given by (5.15), B_5 and B_6 by (5.23) and B_4 is undetermined.

The wave forces on the cylinder are given by (4.9). For consistency the incident wave potential (2.4) must be approximated by its inner expansion. Evaluating this on the cylinder, the only term that contributes to the forces is

$$2iJ_1(\varepsilon)\cos\theta = i\varepsilon\left(1 - \frac{1}{8}\varepsilon^2\right)\cos\theta + O(\varepsilon^5) . \tag{5.25}$$

Using this and the inner scattered potential (5.24) in (4.9) gives the force components

$$f_x = -\frac{\rho g a A}{k} \tanh kh\, \pi\varepsilon \left\{2i - i\varepsilon^2 \ln \varepsilon - \frac{1}{2}\varepsilon^2[\pi(1+\alpha_{11}) + i(1+2C-2\ln 2)] + o(\varepsilon^2)\right\} \tag{5.26}$$

and

$$f_y = \frac{\rho g a A}{k} \tanh kh \left\{\frac{\pi^2 i\varepsilon^3}{4}\beta_{01} + o(\varepsilon^3)\right\}. \tag{5.27}$$

Most of the terms in (5.26) may be accounted for by the expansions in $\varepsilon = ka$ of the exact open-water result (4.11). The in-line force ratio is calculated by taking the expansion in ε of the open-water result which gives

$$F_x = f_x/f^{(o)} = 1 + \varepsilon^2 \frac{\pi i}{4}\alpha_{11} + o(\varepsilon^2). \tag{5.28}$$

Similarly, the force ratio for the cross-channel direction is

$$F_y = f_y/f^{(o)} = -\frac{\varepsilon^2\pi}{8}\beta_{01}. \tag{5.29}$$

The coefficients α_{11} and β_{01} depend on $\mu = kb$ so that calculation of the limit of the force ratios, as the waves become long relative to all other lengths, requires careful consideration. The substitution given by equation (A2) into the definition of α_{11} (3.20) leads to the approximation for small μ

$$\begin{aligned}\alpha_{11} &\approx -\frac{i}{\pi\mu^2}\int_0^\infty \frac{e^{-s} + \cosh(sd/b)}{\sinh s}\, s\, ds \\ &= -\frac{i}{4\pi\mu^2}(2\zeta[2,1] + \zeta[2,(1-d/b)/2] + \zeta[2,(1+d/b)/2])\end{aligned} \tag{5.30}$$

by Gradshteyn and Ryzhik [22, equation 3.552], where ζ is the Riemann zeta function. For a centrally-placed body this reduces to

$$\alpha_{11} \approx -\frac{\pi i}{3\mu^2} \tag{5.31}$$

which from (5.28) gives a long wave limit for the in-line force ratio of

$$F_x = 1 + \frac{\pi^2 a^2}{12 b^2}. \tag{5.32}$$

It should be remembered that it was initially assumed that $a \ll b$ so that (5.32) is not an exact result. A similar calculation shows that, for arbitrary d, $\beta_{01} = O(\mu^{-1})$ as $\mu \to 0$ so that from (5.29) the transverse force ratio F_y is zero in the long-wave limit (for a centrally placed cylinder, $d = 0$, F_y is identically zero for all wavelengths).

6. Plane wave approximation of channel multipoles

A solution method will be given for the problem of scattering by a general axisymmetric body based on a wide spacing approximation. As a first step approximations are derived for the multipole potentials valid for $\mu = kb \gg 1$.

Define

$$\bar{\phi}_n = \phi_n - H_n(kr)\cos n\theta \tag{6.1}$$

where ϕ_n is given by equation (3.7). Each $\bar{\phi}_n$ may be thought of as describing the image set for the corresponding fundamental singular solution. The aim is to obtain an approximation to $\bar{\phi}_n$ for large μ that is valid near the body. The cases n even and odd will be considered separately. For n even put $n = 2N$ then from Gradshteyn and Ryzhik [22, equations 1.331]

$$\cosh 2N\tau = (-1)^N Q(t)\,, \tag{6.2}$$

where

$$Q(t) = (1 - 2N^2t^2 + \dots) \tag{6.3}$$

is a polynomial in t^2 of degree N. By considering the even and odd parts of the integrand, $\bar{\phi}_{2N}$ may be written as a semi-infinite integral

$$\bar{\phi}_{2N} = \frac{1}{\pi i}\int_0^\infty (\mathrm{e}^{\gamma Y}G_1(\gamma) + \mathrm{e}^{-\gamma Y}G_2(\gamma))\cos Xt\,\frac{Q(t)}{\gamma}\,\mathrm{d}t\,, \tag{6.4}$$

where

$$X = R\cos\theta = kx \text{ and } Y = R\sin\theta = k(y-d) \tag{6.5}$$

and

$$G_1(\gamma) = \frac{\mathrm{e}^{-2\gamma\mu} + \mathrm{e}^{2\gamma\nu}}{\sinh 2\gamma\mu}\,,\quad G_2(\gamma) = \frac{\mathrm{e}^{-2\gamma\mu} + \mathrm{e}^{-2\gamma\nu}}{\sinh 2\gamma\mu}\,. \tag{6.6}$$

Make the change of integration variable

$$t = \mathrm{e}^{-i\pi/4}\left\{\frac{s}{\mu}\left(1 + \frac{is}{4\mu}\right)\right\}^{1/2},\quad \gamma = -i\left(1 + \frac{is}{2\mu}\right). \tag{6.7}$$

and deform the contour onto the real x axis in the way described in appendix A to get

$$\bar{\phi}_{2N} = \frac{1}{2\mu\pi i}\int_0^\infty (\mathrm{e}^{\gamma Y}G_1(\gamma) + \mathrm{e}^{-\gamma Y}G_2(\gamma))\cos Xt\,\frac{Q(t)}{t}\,\mathrm{d}s\,. \tag{6.8}$$

For convergence of the integral the restriction $|Y| < 2(\mu - \nu)$, or $|y - d| < 2(b - d)$, is imposed. Now, from (6.7), for $s < 4\mu$

$$t = \mathrm{e}^{-i\pi/4}\left(\frac{s}{\mu}\right)^{1/2}\left(1 + \frac{is}{8\mu} + O(s^2\mu^{-2})\right) \tag{6.9}$$

so that for X, $Y = O(1)$ each term in the integrand of (6.8), with the exceptions of $G_1(\gamma)$ and $G_2(\gamma)$, may be expanded in powers of s/μ. It is shown in Appendix B that the part of the integral over the range 4μ to ∞ is exponentially small as $\mu \to \infty$, so this expansion procedure leads to

$$\bar{\phi}_{2N} = \frac{e^{-i\pi/4}}{2\pi\mu^{1/2}} \left\{ e^{-iY} \int_0^{4\mu} \frac{G_1(\gamma)}{s^{1/2}} \left[1 + \frac{s}{2\mu} \left(Y + i\left(X^2 + 4N^2 - \frac{1}{4} \right) \right) \right] ds \right.$$
$$\left. + e^{iY} \int_0^{4\mu} \frac{G_2(\gamma)}{s^{1/2}} \left[1 + \frac{s}{2\mu} \left(-Y + i\left(X^2 + 4N^2 - \frac{1}{4} \right) \right) \right] ds + O(\mu^{-2}) \right\} . \tag{6.10}$$

A number of the terms in (6.10) may be identified immediately as plane waves propagating in the positive and negative y directions. For the purpose of force calculations the remaining terms may be replaced by equivalent plane waves. The potential on the body, needed for force calculations, may be expressed as a Fourier series in the angle θ, defined in equations (2.1). The vertical force on an axisymmetric body depends only on the constant terms in the series and the horizontal forces depend only on the $\sin\theta$ and $\cos\theta$ terms. Equivalent plane waves are chosen to have the same constant and first harmonic terms as the non-plane terms in (6.10). Now from Abramowitz and Stegun [17, equations (9.1.42–43)],

$$e^{-iY}(Y + iX^2) = \left\{ \sum_{m=0}^{\infty} \left(\varepsilon_m J_{2m}(R) \cos 2m\theta - 2iJ_{2m+1}(R) \sin(2m+1)\theta \right) \right\}$$
$$\times \left\{ R \sin\theta + \frac{iR^2}{2} \left(1 + \cos 2\theta \right) \right\} \tag{6.11}$$

so that the constant and first harmonic components can be isolated using elementary trigonometric identities. Of these, the only non-zero term is $2J_1(R)\sin\theta$ which is the first harmonic term in the expansion of $\sin Y$. Thus

$$e^{-iY}(Y + iX^2) \equiv \sin Y \tag{6.12}$$

in the sense that they will both lead to the same wave forces on an axisymmetric body. Inserting (6.12), and a similar expression with Y replaced by $-Y$, into (6.10) gives the equivalent plane-wave representation

$$\bar{\phi}_{2N} = \frac{e^{-i\pi/4}}{2\pi\mu^{1/2}} \left\{ I_{01} e^{-iY} + I_{02} e^{iY} + \frac{1}{2\mu} \left[I_{11}\left(i\left(4N^2 - \frac{1}{4} \right) e^{-iY} + \sin Y \right) \right. \right.$$
$$\left. \left. + I_{12}\left(i\left(4N^2 - \frac{1}{4} \right) e^{iY} - \sin Y \right) \right] + O(\mu^{-2}) \right\} , \tag{6.13}$$

where

$$I_{pq} = \int_0^{4\mu} s^{p-1/2} G_q(\gamma)\, ds . \tag{6.14}$$

Note that the effect of the images is $O(\mu^{-1/2})$. The integrals in (6.14) are similar to those used by Yeung and Sphaier [4, 5] to help in summing over the images.

Similar results follow for the remaining multipoles. The procedures follow closely that detailed above so only the main results will be given. For n odd put $n = 2N + 1$ then the expansion for large μ is

$$\bar{\phi}_{2N+1} = -\frac{(2N+1)\,\mathrm{e}^{i\pi/4}}{2\pi\mu^{3/2}}\left\{\mathrm{e}^{-iY}X\int_0^{4\mu} s^{1/2}G_1(\gamma)\,\mathrm{d}s + \mathrm{e}^{iY}X\int_0^{4\mu} s^{1/2}G_2(\gamma)\,\mathrm{d}s + O(\mu^{-1})\right\}. \tag{6.15}$$

For the purposes of force calculations

$$\mathrm{e}^{-iY}X \equiv \sin X, \tag{6.16}$$

thus the equivalent plane-wave representation is

$$\bar{\phi}_{2N+1} = -\frac{(2N+1)\,\mathrm{e}^{i\pi/4}}{2\pi\mu^{3/2}}(I_{11} + I_{12})\sin X + O(\mu^{-5/2}). \tag{6.17}$$

For the second set of multipoles define

$$\bar{\psi}_n = \psi_n - H_n(kr)\sin n\theta, \tag{6.18}$$

where ψ_n is given by equation (3.14). For even $n = 2N$, the expansion for large μ is

$$\bar{\psi}_{2N} = -\frac{N\,\mathrm{e}^{3i\pi/4}}{\pi\mu^{3/2}}\left\{\mathrm{e}^{-iY}X\int_0^{4\mu} s^{1/2}G_3(\gamma)\,\mathrm{d}s - \mathrm{e}^{iY}X\int_0^{4\mu} s^{1/2}G_4(\gamma)\,\mathrm{d}s + O(\mu^{-1})\right\} \tag{6.19}$$

and, using (6.16), this has the equivalent plane-wave representation

$$\bar{\psi}_{2N} = -\frac{N\,\mathrm{e}^{3i\pi/4}}{\pi\mu^{3/2}}(I_{13} - I_{14})\sin X + O(\mu^{-5/2}), \tag{6.20}$$

where I_{pq} is defined by (6.14) and

$$G_3(\gamma) = \frac{\mathrm{e}^{-2\gamma\mu} - \mathrm{e}^{2\gamma\nu}}{\sinh 2\gamma\mu}, \quad G_4(\gamma) = \frac{\mathrm{e}^{-2\gamma\mu} - \mathrm{e}^{-2\gamma\nu}}{\sinh 2\gamma\mu}. \tag{6.21}$$

Finally, for odd $n = 2N + 1$, the expansion for large μ is

$$\begin{aligned}\bar{\psi}_{2N+1} = \frac{\mathrm{e}^{i\pi/4}}{2\pi\mu^{1/2}}\Bigg\{&\mathrm{e}^{-iY}\int_0^{4\mu}\frac{G_3(\gamma)}{s^{1/2}}\left[1 + \frac{s}{2\mu}\left(Y + i\left(X^2 + 4N(N+1) + \frac{3}{4}\right)\right)\right]\mathrm{d}s \\ &- \mathrm{e}^{iY}\int_0^{4\mu}\frac{G_4(\gamma)}{s^{1/2}}\left[1 + \frac{s}{2\mu}\left(-Y + i\left(X^2 + 4N(N+1) + \frac{3}{4}\right)\right)\right]\mathrm{d}s + O(\mu^{-2})\Bigg\}\end{aligned} \tag{6.22}$$

which has the equivalent plane-wave representation

$$\begin{aligned}\bar{\psi}_{2N+1} = \frac{\mathrm{e}^{i\pi/4}}{2\pi\mu^{1/2}}\Bigg\{&I_{03}\,\mathrm{e}^{-iY} - I_{04}\,\mathrm{e}^{iY} + \frac{1}{2\mu}\left[I_{13}\left(i\left(4N(N+1) + \frac{3}{4}\right)\mathrm{e}^{-iY} + \sin Y\right)\right. \\ &\left. - I_{14}\left(i\left(4N(N+1) + \frac{3}{4}\right)\mathrm{e}^{iY} - \sin Y\right)\right] + O(\mu^{-2})\Bigg\}.\end{aligned} \tag{6.23}$$

7. Plane wave approximation for scattering by axisymmetric bodies

The scattering of a plane wave by a general axisymmetric body in a channel is now considered. The boundary value problem to be solved is similar to that described in Section 2 for the vertical cylinders. The only point of difference is that the body boundary condition is generalized to

$$\frac{\partial \phi}{\partial n} = -\frac{\partial \phi_I}{\partial n} \tag{7.1}$$

on the body surface, where n is a coordinate measured normal to the body.

Sufficiently far from the body for evanescent modes to be negligible, the scattered wave field has the following expansion in terms of the channel multipoles derived in section 3:

$$\phi = \sum_{n=0}^{\infty} \varepsilon_n i^n (A_n \phi_n + B_n \psi_n) . \tag{7.2}$$

For the vertical cylinder, treated in Section 4, there are no evanescent modes and this representation is exact. Introducing the decomposition of the multipoles given in equations (6.1) and (6.18), equation (7.2) may be written

$$\phi = \sum_{n=0}^{\infty} \varepsilon_n i^n H_n(kr)(A_n \cos n\theta + B_n \sin n\theta) + \bar{\phi} , \tag{7.3}$$

where

$$\bar{\phi} = \sum_{n=0}^{\infty} \varepsilon_n i^n (A_n \bar{\phi}_n + B_n \bar{\psi}_n) . \tag{7.4}$$

As before a fictitious multipole ψ_0 has been retained for convenience so that there is the implicit assumption that $B_0 = 0$. The non-singular parts of the multipole potentials, $\bar{\phi}_n$ and $\bar{\psi}_n$, are the corrections to the open water multipoles resulting from the presence of the channel walls. Thus, $\bar{\phi}$ may be thought of as the total wave field incident upon the body as a result of all scattering by the images in the walls. Each image scatters the incident wave (2.4) and the waves emanating from the remaining images and the body itself. In turn, $\bar{\phi}$ and the incident wave (2.4) are scattered by the body to give the outgoing waves described by the explicit summation in equation (7.3).

For $\mu \gg 1$, so that the channel walls are widely spaced compared to the wavelength, the non-singular parts of the multipole potentials can be approximated using equations (6.13), (6.17), (6.20) and (6.23) to obtain

$$\bar{\phi} = (C_1 + D_1)\, e^{-ik(y-d)} + (C_2 + D_2)\, e^{ik(y-d)} + D_3 \sin kx + O(\mu^{-5/2}) , \tag{7.5}$$

where

$$C_1 = S \sum_{n=0}^{\infty} (-1)^n (\varepsilon_n A_{2n} I_{01} - 2B_{2n+1} I_{03}) , \tag{7.6}$$

$$C_2 = S \sum_{n=0}^{\infty} (-1)^n (\varepsilon_n A_{2n} I_{02} + 2B_{2n+1} I_{04}) , \tag{7.7}$$

$$D_1 = \frac{iS}{2\mu} \sum_{n=0}^{\infty} (-1)^n \left[\varepsilon_n A_{2n}\left(\left(4n^2 + \frac{1}{4}\right)I_{11} - \frac{1}{2} I_{12}\right) - 2B_{2n+1}\left(\left(4n(n+1) + \frac{5}{4}\right)I_{13} + \frac{1}{2} I_{14}\right)\right], \tag{7.8}$$

$$D_2 = \frac{iS}{2\mu} \sum_{n=0}^{\infty} (-1)^n \left[\varepsilon_n A_{2n}\left(\left(4n^2 + \frac{1}{4}\right)I_{12} - \frac{1}{2} I_{11}\right) + 2B_{2n+1}\left(\left(4n(n+1) + \frac{5}{4}\right)I_{14} + \frac{1}{2} I_{13}\right)\right], \tag{7.9}$$

$$D_3 = \frac{2S}{\mu} \sum_{n=0}^{\infty} (-1)^n \left[(2n+1)A_{2n+1}(I_{11} + I_{12}) + 2nB_{2n}(I_{13} - I_{14})\right] \tag{7.10}$$

and

$$S = \frac{e^{-i\pi/4}}{2\pi\mu^{1/2}} . \tag{7.11}$$

Note that $C_j = O(\mu^{-1/2})$, $j = 1, 2$ and $D_j = O(\mu^{-3/2})$, $j = 1, 2, 3$. To leading order, the waves scattered by the image bodies appear as plane waves propagating along the line of the images when observed in the vicinity of the original body. So to a first approximation the body can be regarded as being in open water, that is with no channel walls, under the action of a total incident wave

$$\bar{\phi}\gamma \sim e^{ikx} + C_1 e^{-ik(y-d)} + C_2 e^{ik(y-d)} = e^{ikr\cos\theta} + C_1 e^{ikr\cos(\theta+\pi/2)} + C_2 e^{ikr\cos(\theta-\pi/2)} . \tag{7.12}$$

Suppose that the scattering properties of the body when in open water are known. That is, for an incident wave e^{ikx}, sufficiently far from the body for evanescent modes to be negligibly small, the solution for the scattered field may be written

$$\phi^{(o)} = \sum_{n=0}^{\infty} \varepsilon_n i^n A_n^{(o)} H_n(kr) \cos n\theta , \tag{7.13}$$

where the open water scattering coefficients $\{A_n^{(o)}, n = 0, 1, 2 \ldots\}$ are known. The scattered field due to the incident wave (7.12) will be

$$\phi - \bar{\phi} = \sum_{n=0}^{\infty} \varepsilon_n i^n A_n^{(o)} H_n(kr)(\cos n\theta + C_1 \cos n(\theta + \pi/2) + C_2 \cos n(\theta - \pi/2)) . \tag{7.14}$$

Comparing (7.14) with (7.3) gives

$$A_n = A_n^{(o)}\left[1 + (C_1 + C_2)\cos\frac{n\pi}{2}\right] \tag{7.15}$$

and

$$B_n = -A_n^{(o)}(C_1 - C_2)\sin\frac{n\pi}{2} . \tag{7.16}$$

Substitution of (7.15–16) into (7.6–7) and rearranging gives the simultaneous equations

$$C_1\left[1 - S\left(I_{01} \sum_{n=0}^{\infty} \varepsilon_n A_{2n}^{(o)} + 2I_{03} \sum_{n=0}^{\infty} A_{2n+1}^{(o)}\right)\right] - C_2 S\left(I_{01} \sum_{n=0}^{\infty} \varepsilon_n A_{2n}^{(o)} - 2I_{03} \sum_{n=0}^{\infty} A_{2n+1}^{(o)}\right)$$
$$= SI_{01} \sum_{n=0}^{\infty} (-1)^n \varepsilon_n A_{2n}^{(o)} \tag{7.17}$$

and

$$C_2\left[1 - S\left(I_{02} \sum_{n=0}^{\infty} \varepsilon_n A_{2n}^{(o)} + 2I_{04} \sum_{n=0}^{\infty} A_{2n+1}^{(o)}\right)\right] - C_1 S\left(I_{02} \sum_{n=0}^{\infty} \varepsilon_n A_{2n}^{(o)} - 2I_{04} \sum_{n=0}^{\infty} A_{2n+1}^{(o)}\right)$$
$$= SI_{02} \sum_{n=0}^{\infty} (-1)^n \varepsilon_n A_{2n}^{(o)} \tag{7.18}$$

which may be readily solved for C_1 and C_2.

The scattering of the correction terms, the waves with amplitude D_j, has been neglected in calculating the primary wave amplitudes C_j. Thus, in equations (7.17–18), terms of $O(\mu^{-2})$ have been neglected giving errors of the same magnitude in C_1 and C_2. The correction amplitudes D_j can be calculated to the same order of accuracy by neglecting terms of $O(\mu^{-1/2})$ in the scattering coefficients of (7.15–16), that is, by using $A_n \sim A_n^{(o)}$ and $B_n \sim 0$ in (7.8–10).

The wave forces on the body follow very simply from (7.5). If $f^{(o)}$ is the in-line first-order force in open water then, allowing for the phases of the different waves in (7.5), the x and y-components are respectively

$$f_x = f^{(o)}(1 - iD_3) \text{ and } f_y = f^{(o)}(-C_1 - D_1 + C_2 + D_2)\,. \tag{7.19}$$

8. Results

All of the results for forces given in this section are scaled by the in-line force on that body when it is placed in open water. To keep the amount of data manageable, results are given only for horizontal components of the force. Results for vertical forces on a truncated cylinder may be found in Yeung and Sphaier [4] and truncated and full-cylinder reflection coefficients in Linton and Evans [10]. Both of these papers consider the case of a centrally-placed body only.

The results in Figs 2–9 are all for a vertical cylinder extending throughout the depth and in each figure comparison is made between the accurate solution method described in Section 4, the small-body solution of Section 5 and the plane-wave approximation of Section 7. The small-body solution assumes that the cylinder radius a is much less than the channel width b and that $kb = O(1)$, where k is the wavenumber. Taken together these assumptions imply $ka \ll 1$. The fundamental assumption behind the plane-wave approximation was expressed as $kb \gg 1$ but, more exactly, it is assumed that the distance between the body and its images in the channel walls is much smaller than the wavelength. In terms of the geometrical parameters in Fig. 1, this may be expressed as $k(b - (a + |d|)) \gg 1$.

In Fig. 2 the modulus of the in-line force F_x, defined by equations (4.9–12), is given as a function of wave-number when the cylinder is on the centre-line of the channel. For Fig.

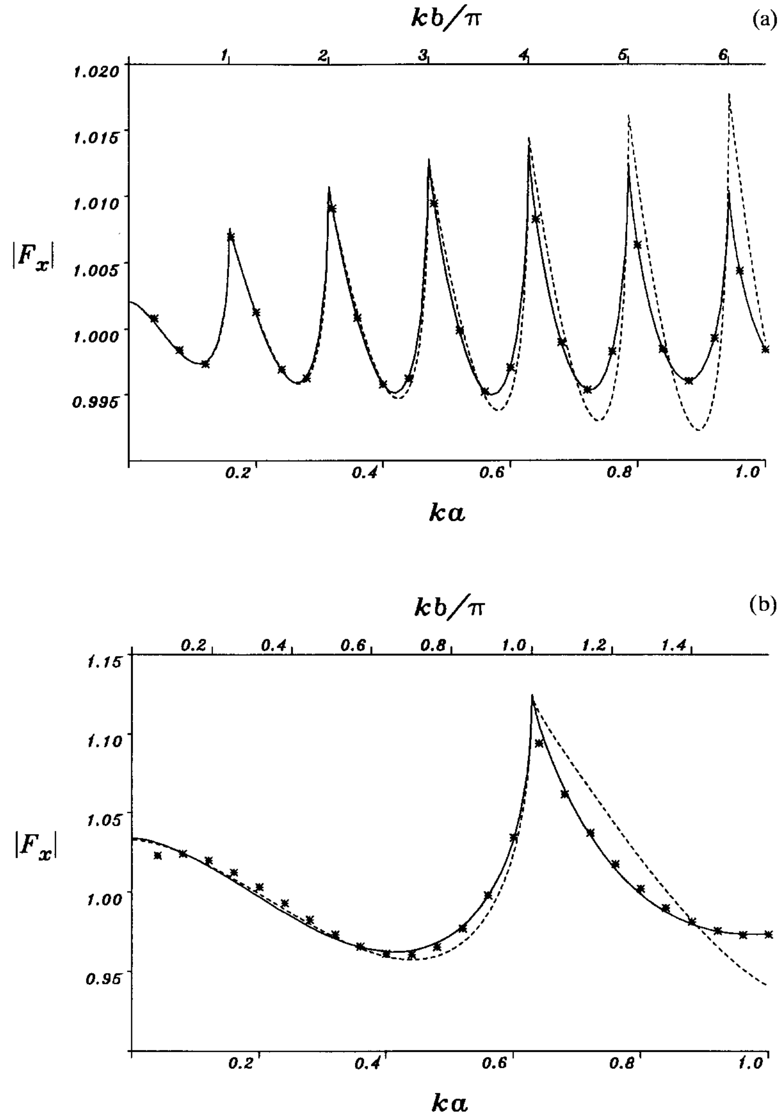


Fig. 2. In-line force ratio $|F_x|$ v. ka. Comparison of full theory (——) with small-body approximation (---) and plane-wave approximation (***) for $d/b=0$ and (a) $b/a=20$, (b) $b/a=5$.

(2a), $b/a=20$ and there are resonant peaks in the curve at about $kb=n\pi$, or $ka=n\pi/20$ where n is an integer, corresponding to poles of the integrand in equation (3.8) or, in physical terms, to standing waves across the channel in the absence of the body. For all cylinders of non-zero radius, detuning occurs and the resonant peaks in the force curves are of finite height with the maxima not coinciding exactly with the cross-channel standing wave frequencies $kb=n\pi$. With the cylinder on the centre-line of the channel only the standing-waves symmetric in y give rise to peaks in the force curves. For fig. 2(b), $b/a=5$ and there is only a single peak in the range of ka given. In both of these figures, the plane wave method performs well for all except the very longest waves, down to values of $kb \sim 0.5$, despite the fact that the plane-wave method was derived under the assumption $kb \gg 1$. As might be anticipated, the better agreement is obtained for the wider channel relative to the cylinder radius when the body is far from its images. Note that without the correction term in (7.19) the plane-wave method gives $F_x = f_x/f^{(o)} \equiv 1$ for all frequencies. The small-body solution is complementary to the plane-wave approximation in that it performs best for long waves. As the long-wave limit is approached, the results are indistinguishable from the full solution for $b/a = 20$ but there are minor discrepancies for $b/a = 5$. This is not surprising as the small-body solution was derived under the assumption that $b \gg a$.

In Fig. 3 are results for in-line (x) and cross-channel (y) forces when the cylinder is offset from the channel centre line by a distance $d = 0.25b$ with $b/a = 5$. Comparison of Fig. 3(a) with Fig. 2(b) shows that there are additional peaks roughly corresponding to the antisymmetric standing waves at $kb = (n + 1/2)\pi$. With the offset cylinder the height of the peak

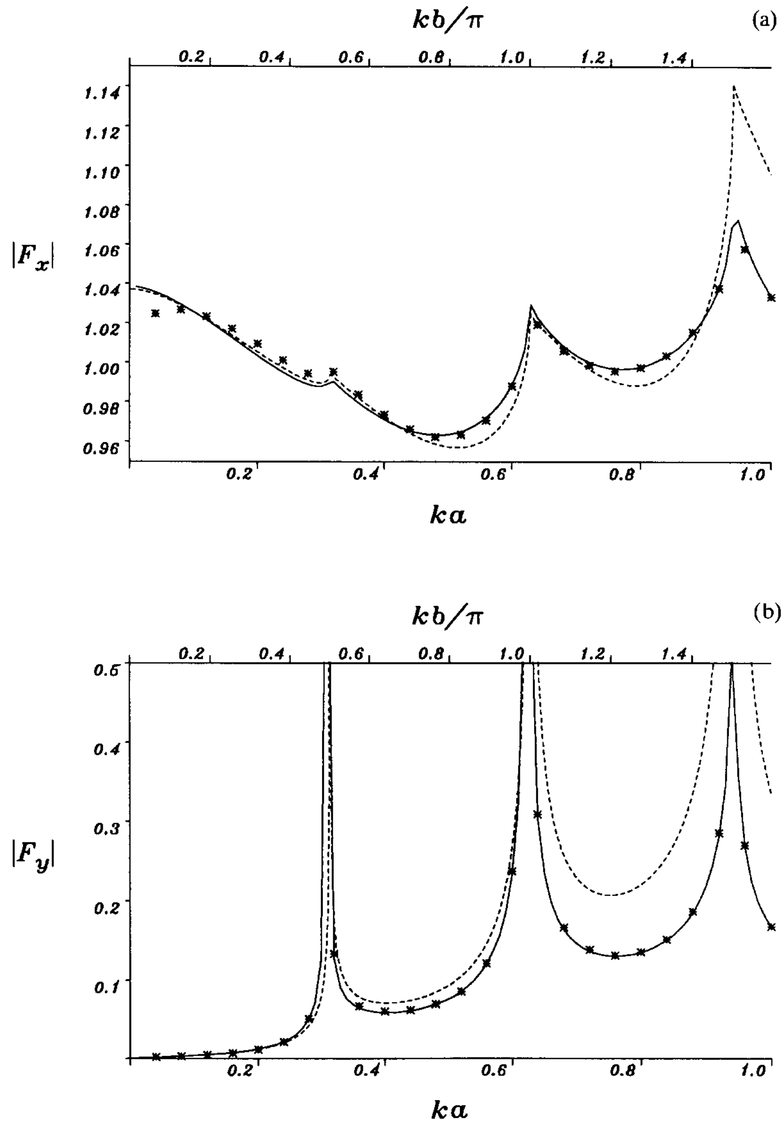


Fig. 3. Modulus of force ratio v. ka. Comparison of full theory (——) with small-body approximation (---) and plane-wave approximation (***) for $b/a = 5$, $d/b = 0.25$. (a) In-line force $|F_x|$, (b) cross-channel force $|F_y|$.

corresponding to the symmetric standing wave is much reduced. The cross-channel force, Fig. 3(b), displays very strong peaks though they remain finite.

For a fixed circular cylinder on the centre plane of the channel, trapped wave solutions exist [8, 9]. These are free oscillations of the fluid that decay along the channel (and so are of finite energy) and which are anti-symmetric about the channel centre plane. The trapped-wave solutions exist at discrete frequencies that depend on the cylinder radius and the channel width. The existence of these solutions leads to characteristic resonances for an antisymmetric forced motion, for example they are observed if the cylinder is forced to sway in the cross-channel direction [10]. The scattering problem is symmetric about the channel centre plane and therefore no resonance is observed at a trapping frequency. There is no evidence that trapped-wave solutions exist for cylinders offset from the channel centre plane and this is supported by the lack of unexplained resonances in Fig. 3.

In Figs 4–7, results are given for offset cylinders with the wave forces given as a function of the off-centre distance d/b for fixed wavenumber ka and channel width b/a. Figures 4–5

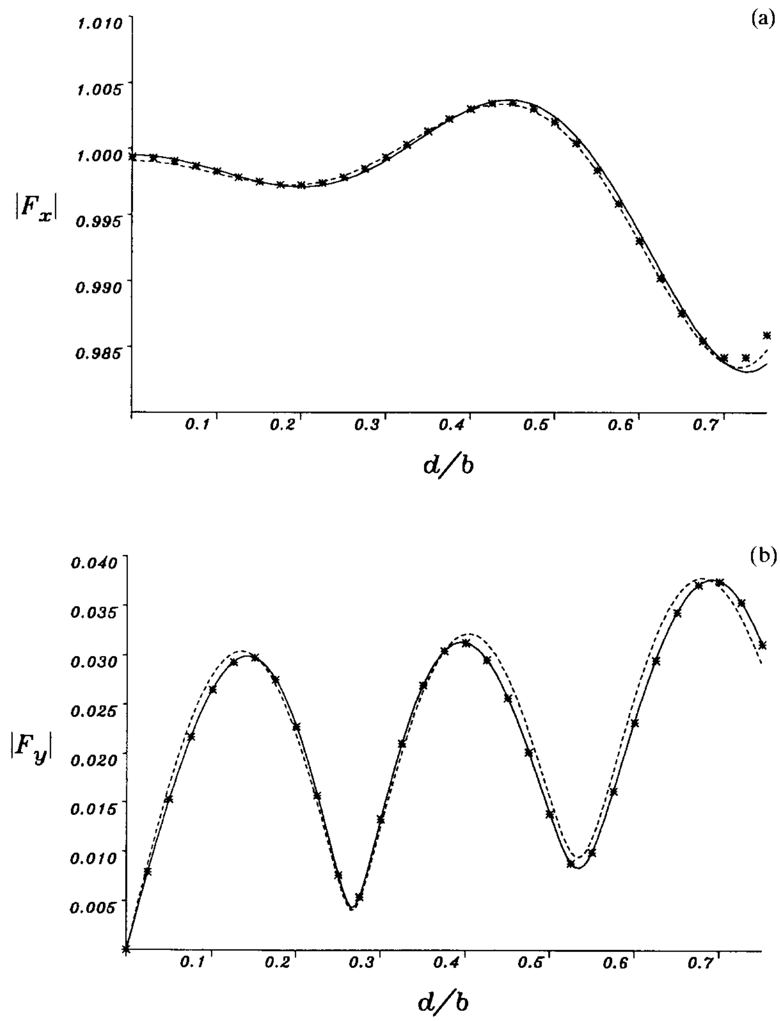
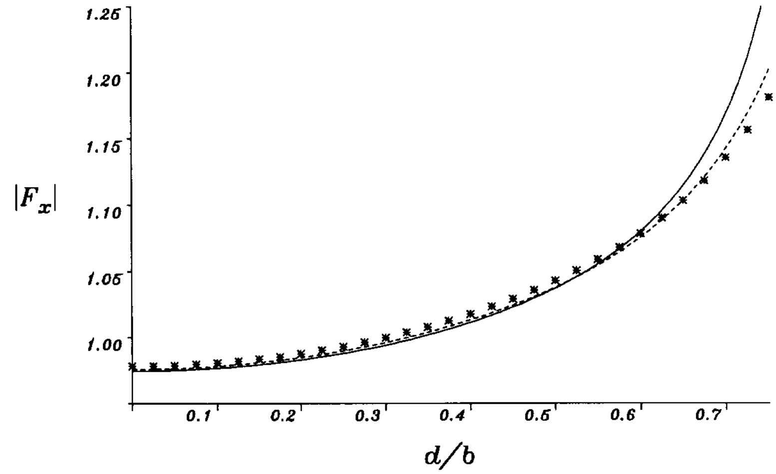
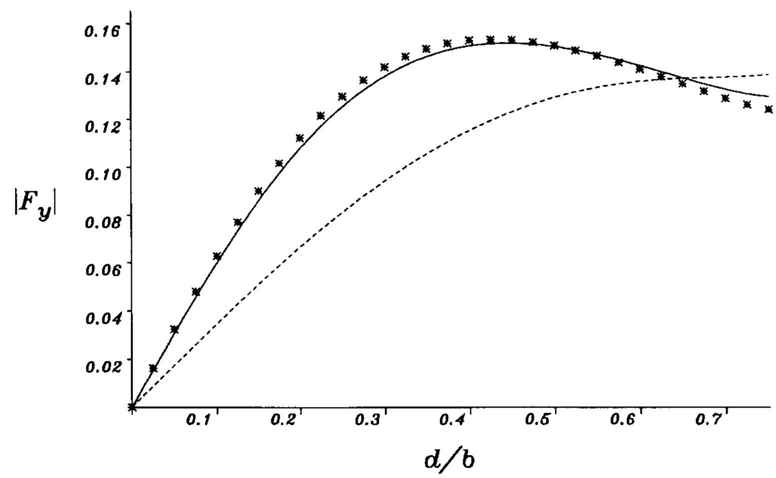


Fig. 4. Modulus of force ratio v. d/b. Comparison of full theory (——) with small-body approximation (---) and plane-wave approximation (***) for $b/a = 20$, $ka = 0.3$. (a) In-line force $|F_x|$, (b) cross-channel force $|F_y|$.



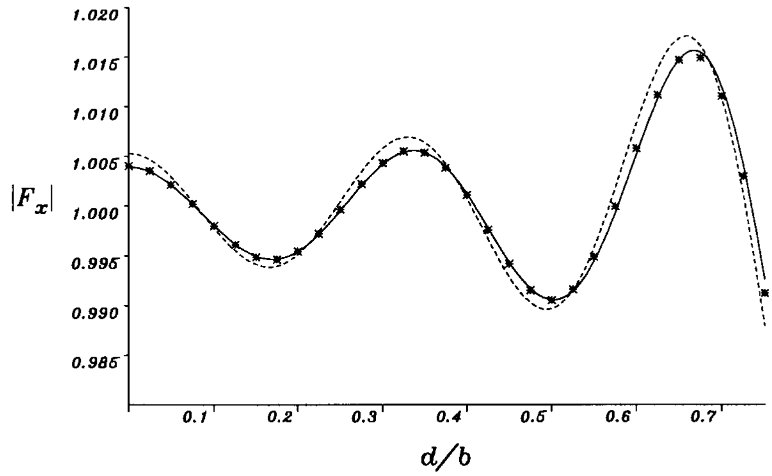
(a)



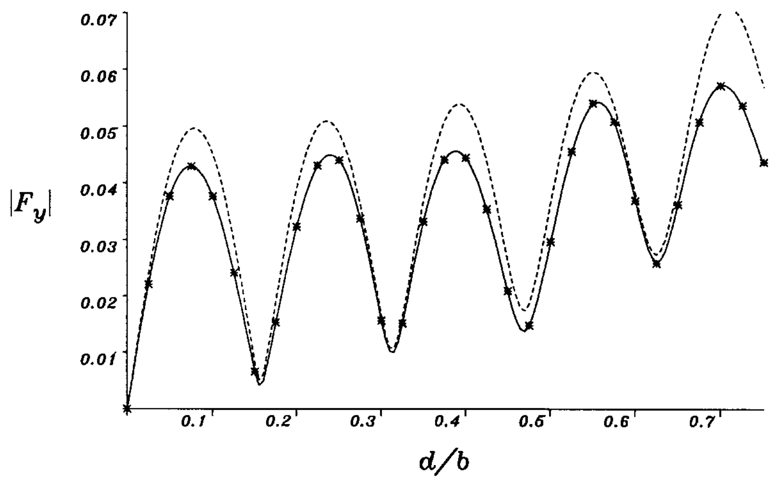
(b)

Fig. 5. Modulus of force ratio v. d/b. Comparison of full theory (——) with small-body approximation (---) and plane-wave approximation (***) for $b/a = 5$, $ka = 0.3$. (a) In-line force $|F_x|$, (b) cross-channel force $|F_y|$.

have $ka = 0.3$, which is close to the first antisymmetric standing wave resonance, while Figs 6–7 have $ka = 0.5$, which is not close to any resonance. It might be anticipated that the approximate solutions will perform less well near a resonance. However, both approximate solutions maintain good accuracy except for the small-body solution in Fig. 5(b). Away from resonance, the corresponding results show an improvement in accuracy. It is noteworthy that the plane-wave method consistently gives a better approximation to the x-force than the y-force. Both x and y components of the force oscillate with the off-centre distance but for all sets of parameters the y-force is the more oscillatory. This is presumably the result of standing waves set up between the body and channel walls at certain offsets. At the higher frequency, Figs 6–7, there are more oscillations.



(a)



(b)

Fig. 6. Modulus of force ratio v. d/b. Comparison of full theory (——) with small-body approximation (–––) and plane-wave approximation (***) for $b/a = 20$, $ka = 0.5$. (a) In-line force $|F_x|$, (b) cross-channel force $|F_y|$.

The remaining results, Figs 8–9, are for a truncated vertical cylinder of draught D floating in water of depth h. For this set of results the values $b/a = 20$ and $b/h = 1$ are used throughout. All calculations are by the plane-wave method described in Section 7 which requires knowledges of the scattering coefficients of the body, defined by equation (7.13), when in open water. These coefficients may be calculated using the method of Garrett [23]. Figure 8 gives the in-line force on a centrally-placed body for three different draughts. As the draught is increased the motion is more sensitive to the channel resonances and the force curves become more peaked. Finally, Fig. 9 gives the x- and y-forces at $ka = 0.3$ as a function of off-centre distance for the same three draughts. The x-force ratio varies most strongly for the deeper draft cylinders but it is noteworthy that the y-forces are larger, relative to the in-line open-sea values, for the shallow draft body.

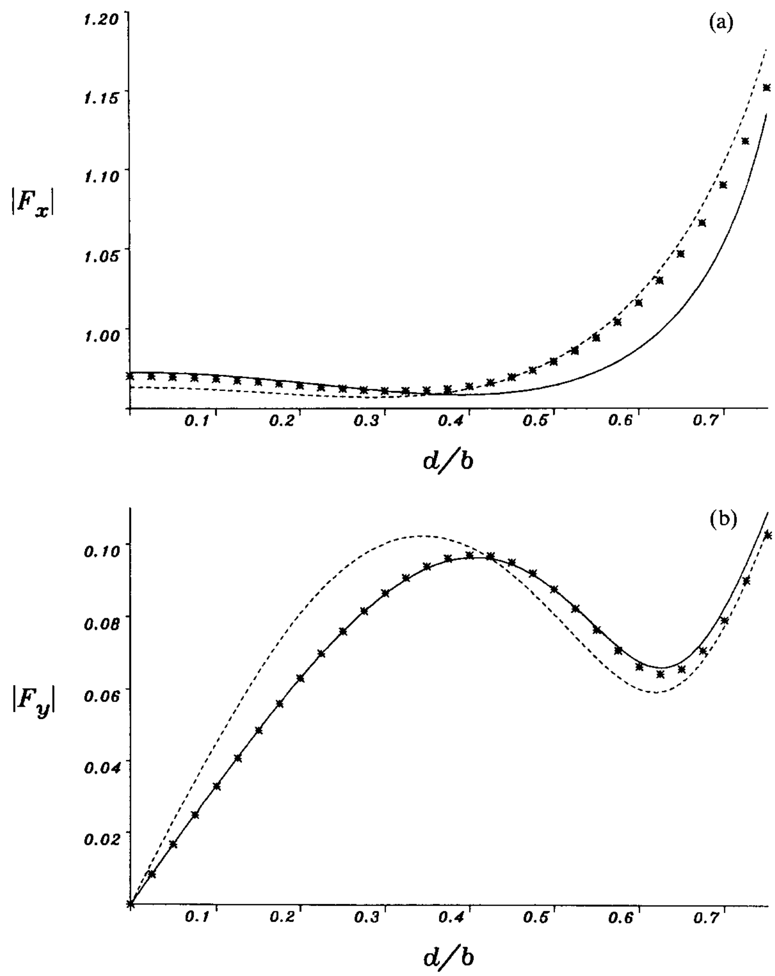


Fig. 7. Modulus of force ratio v. d/b. Comparison of full theory (——) with small-body approximation (– – –) and plane-wave approximation (***) for $b/a = 5$, $ka = 0.5$. (a) In-line force $|F_x|$, (b) cross-channel force $|F_y|$.

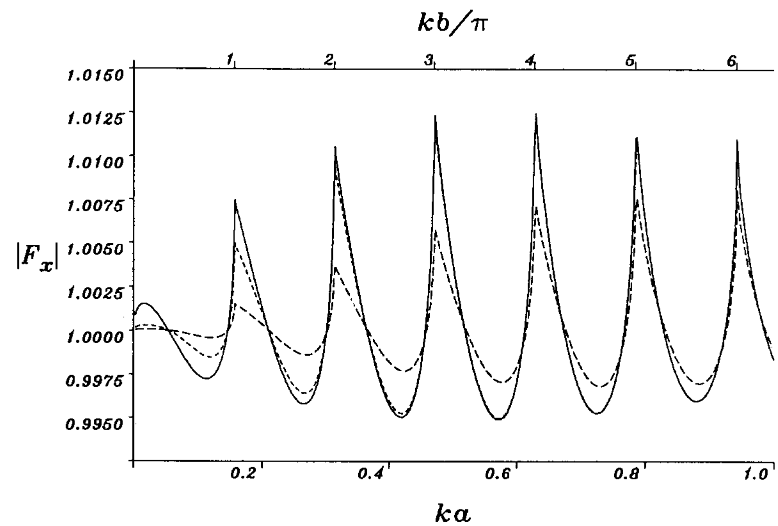


Fig. 8. In-line force ratio $|F_x|$ v. ka for truncated cylinders with draughts $D/h = 1$, (——), $D/h = 0.2$ (---) and $D/h = 0.05$ (– – –) for $b/a = 20$, $d/b = 0$.

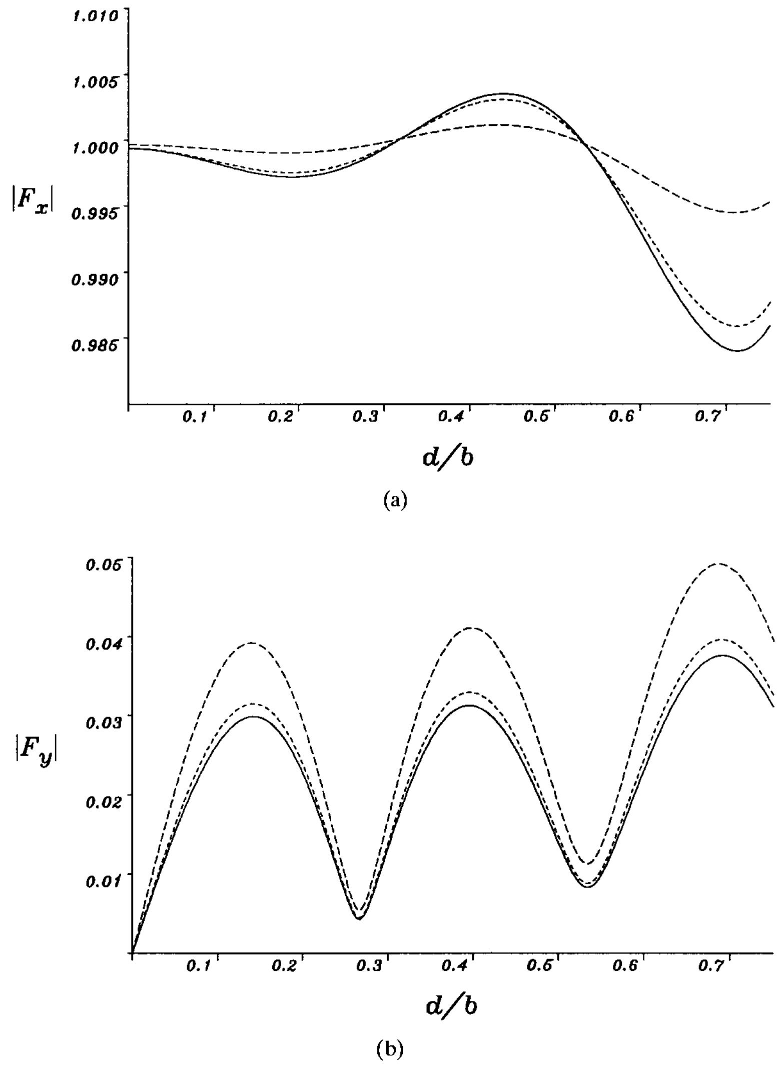


Fig. 9. Modulus of force ratio v. d/b for truncated cylinders with draughts $D/h = 1$ (——), $D/h = 0.2$ (---) and $D/h = 0.05$ (– – –) for $b/a = 20$, $ka = 0.3$. (a) In-line force $|F_x|$, (b) cross-channel force $|F_y|$.

9. Conclusion

A variety of methods have been presented for the treatment of the scattering problem for a vertically axisymmetric body of radius a placed arbitrarily between vertical channel walls a distance b apart. A new 'accurate' formulation for the case of a circular cylinder extending throughout the depth allows easier calculation than previous methods as there is no need to sum series of image singularities. In addition, two approximate solution methods have been given. The first is applicable only to the vertical cylinder and is based on the assumption that the radius is much smaller than other lengths in the problem. Simple explicit expressions were derived for the wave forces on the cylinder and found to agree very well with the accurate solution up to about $ka = 0.5$, where k is the wavenumber. More importantly, this solution describes qualitatively the features of the solution over the whole range of ka likely to be of practical interest. The second approximate solution is based on the assumption that the bodies are widely separated from their images on the scale of the wavelength. This 'plane-wave' approximation may be applied to any axisymmetric body but requires previous knowledge of the scattering properties of the body when in open water. Given this information calculations are quite straightforward, the in-line force requires only summation of a simple series and numerical evaluation of an integral. The plane-wave approximation was compared with the full-vertical cylinder solution and found to give excellent results for all but the very longest waves.

Appendix A: Numerical evaluation of multipole expansion coefficients

The channel multipole expansion coefficients are given by equations (3.20–21) and (3.23–24). As indicated in Section 3, the coefficients with odd integrands are zero. The non-zero coefficients with even integrands may be written as integrals over a semi-infinite range with the modified integration path running beneath any poles in order to satisfy the radiation condition. From Gradshteyn and Ryzhik [22, equations 1.331], each of the products of hyperbolic functions in equations (3.20–21) and (3.23–24) may be expressed in the form $\gamma^p t^q P(t)$, where $P(t)$, $P(0) \neq 0$, is a finite degree polynomial in t^2 and p and q take the values 0, 1 or 2. Each of the expansion coefficients may then be written in the form

$$I = \int_0^\infty F(\gamma)\gamma^{p-1} t^q P(t)\, \mathrm{d}t\,, \tag{A1}$$

where $F(\gamma)$ contains all the exponentials of $2\gamma\mu$ and $2\gamma\nu$. To evaluate the integral in its present form requires careful consideration of the poles on the real axis. To avoid this complication make the change of integration variable

$$t = \mathrm{e}^{-i\pi/4}\left\{\frac{s}{\mu}\left(1 + \frac{is}{4\mu}\right)\right\}^{1/2}, \quad \gamma = -i\left(1 + \frac{is}{2\mu}\right). \tag{A2}$$

In the s-plane all singularities lie on the imaginary axis and, in particular, the poles on the $\Re t$-axis are transformed onto $0 \leq \Im s \leq 2\mu$. The integration path in the s-plane runs from the origin to $2\mu i$ along the imaginary axis, passing to the right of any poles, and then moves off the axis and approaches $s = 2\mu t$ as $t \to \infty$. As all of the poles are on the imaginary axis, the path of integration may now be deformed to run along the positive $\Re s$-axis, which is free of singularities, to give

$$I = \frac{1}{2\mu}\int_0^\infty F(\gamma)\gamma^p t^{q-1} P(t)\, \mathrm{d}s \tag{A3}$$

and this is easily evaluated numerically after first separating into real and imaginary parts.

Appendix B: Integral estimate

Here it is demonstrated that the contribution to the integral in (6.8) from the range $(4\mu, \infty)$ may be neglected. The function $Q(t)$ is a polynomial of degree N in t^2 and therefore, by

(6.7), a polynomial of degree $2N$ in s. Thus, neglecting constant factors independent of μ, the general form of contributions to the first term on the right-hand side of (6.8) is

$$I = \frac{1}{\mu} \int_{4\mu}^{\infty} e^{\gamma Y} G_1(\gamma) \cos Xt \, \frac{s^M}{t} \, ds \, , \; 0 \leqslant M \leqslant 2N \, . \tag{B1}$$

The aim is to obtain a bound for I. First of all, bounds are obtained for the component factors in the integrand. For $s > 0$ and $|Y| < 2(\mu - \nu)$,

$$\begin{aligned} |e^{\gamma Y} G_1(\gamma)| &= \left| e^{sY/2\mu} \left(\frac{e^{-(s-2\mu i)} + e^{(s-2\mu i)\nu/\mu}}{\frac{1}{2}(e^{s-2\mu i} - e^{-(s-2\mu i)})} \right) \right| \leqslant 2e^{sY/2\mu} \, \frac{e^{-s} + e^{s\nu/\mu}}{e^{s} - e^{-s}} \\ &\leqslant \frac{4}{e^{s(1-Y/2\mu-\nu/\mu)} - e^{-s(1+Y/2\mu+\nu/\mu)}} \leqslant \frac{4}{e^{\beta s} - 1} \, . \end{aligned} \tag{B2}$$

where

$$\beta = 1 - \frac{Y}{2\mu} - \frac{\nu}{\mu} > 0 \, . \tag{B3}$$

The first inequality in (B2) uses elementary inequalities for complex numbers and the remaining steps use properties of the exponential function. From (6.7), for real, positive s, $0 \leqslant |\Im t| \leqslant 1$ and so

$$|\cos Xt| \leqslant \frac{1}{2} \left(|e^{iXt}| + |e^{-iXt}| \right) \leqslant e^{|X \Im t|} \leqslant e^{X} \, . \tag{B4}$$

Finally, for $s \geqslant 4\mu$,

$$|t| = \left\{ \frac{s}{\mu} \left(1 + \frac{s^2}{16\mu^2} \right) \right\}^{1/2} \geqslant 2^{5/4} \, . \tag{B5}$$

Using these inequalities in the definition of I, equation (B1), gives

$$|I| \leqslant \frac{K_1}{\mu} \int_{4\mu}^{\infty} \frac{s^M \, ds}{e^{\beta s} - 1} \, , \tag{B6}$$

where K_j, $j = 1, 2, 3$ are constants independent of μ and M. The substitution $s = 4\mu(u + 1)$ gives

$$|I| \leqslant K_2 (4\mu)^M e^{-4\mu\beta} \int_0^{\infty} \frac{(u+1)^M \, du}{e^{4\mu\beta u} - e^{-4\mu\beta}} \, . \tag{B7}$$

The integration in (B7) may be carried out using the result given by Gradshteyn and Ryzhik [22, equation 3.411.22] and elementary inequalities finally give

$$|I| \leqslant K_3 (M+1)! (4\mu)^M \, e^{-4\mu\beta} \, . \tag{B8}$$

A similar inequality holds for the contributions to the second term on the right-hand side of (6.8). Thus, under the assumption that $\mu \gg 1$, the contribution to $\bar{\phi}_{2N}$ from the integration range $(4\mu, \infty)$ is negligible compared to the terms retained in (6.10).

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